Chapter 12. Integrable Highest-weight module over affine algebras

§12.1

$A \longrightarrow X_N^{(r)} \to$ Affr

$X = A, B, C, D, E, F$ or G

$r = 1, 2, 3$

$g(A) \longrightarrow$ affine algebra of type $X_N^{(r)}$

$\alpha \in \Delta$ mult(α)

$(A)_{n \times n} \to$ rank $= l$

Recall denominator identity (10.4.4) $n - l = \dim \mathfrak{h} - n$

$$R: \prod_{\alpha \in \Delta_+} (1 - e(-\alpha))^{\text{mult}\,\alpha} = \sum_{w \in W} \varepsilon(w)\, e(w(\rho) - \rho)$$

Δ rank$(A) = l$ $\{e_i, f_i, i = 1, \dots, l\}$

(Recall prop 6.3).

$X_N^{(1)}$ $\Delta^{re} = \{\alpha + n\delta \mid \alpha \in \mathring{\Delta}, n \in \mathbb{Z}\}$ $-\alpha + n\delta$

mult$(\alpha) = \dim \mathfrak{g}_\alpha$

$A_l^{(1)}$

$$R_{\text{left}} = \prod_{\alpha \in \mathring{\Delta}_+} (1 - e(-\alpha)) \prod_{m=1}^{\infty} (1 - e(-m\delta))^l \prod_{\beta \in \mathring{\Delta}} (1 - e(\beta - m\delta))$$

$\beta + n\delta$

$L(x)$:

$X_l^{(1)}$ $e(-\delta)$ $x = e(-m\delta)$

Δ^{re}

$$L(x) = (1 - x)^l \prod_{\alpha \in \mathring{\Delta}} (1 - x e(\alpha)) \qquad A \in X_N^{(1)}$$

$$L(x) = (1 - x)^s (1 - x^r)^{l-s} \prod_{\alpha \in \mathring{\Delta}_s} (1 - x e(\alpha)) \prod_{\alpha \in \mathring{\Delta}_l} (1 - x^r e(\alpha))$$

$A \in X_N^{(2)}$ or $X_N^{(3)}$ $A \neq A_{2l}^{(2)}$

$\mathring{\Delta}_+$

$\Delta^{re} = \{\alpha + n\delta \mid \alpha \in \mathring{\Delta}_s, n \in \mathbb{Z}\} \cup \{\alpha + nr\delta \mid \alpha \in \mathring{\Delta}_l, n \in \mathbb{Z}\}$

$r = 2$ or 3 $A \neq A_{2l}^{(2)}$

$\Delta_+^{re} \cup \delta_+ \cup$ 虚根

$$\checkmark\ L(x) = (1-x)^l \prod_{\alpha\in\mathring{\Delta}_s} (1 - x e(\alpha)) \prod^{\circ}_{\alpha\in\Delta_l} (1-xe(\tfrac{1}{2}\delta-\alpha))(1-x^2e(\alpha)))$$

$$A = A^{(2)}_{2l}$$

$$\mathring{R} = \prod_{\alpha\in\mathring{\Delta}_+} (1-e(-\alpha))$$

$$R_{\text{left}} = \mathring{R}\prod_{n\geq 1} L(e(n\delta))\ \checkmark$$

$u\in\mathring{W}$, t_α $\alpha\in M$

$$R: \prod_{\alpha\in\Delta_+}(1-e(-\alpha))^{\text{mult}\,\alpha} = \sum_{w\in W}\varepsilon(w)e(\overset{ut_\alpha}{w}(\rho)-\rho)$$

Δ $\checkmark$

R

- $\rho \longrightarrow$ §6.2 $\rho\in\mathfrak{h}^*$ by $\langle\rho, \alpha_i^\vee\rangle = 1$ $(i=0,\dots,l)$

$$\langle\rho, d\rangle = 0 \quad \langle\rho,\alpha_i^\vee\rangle = 1$$

$$\langle\rho, K\rangle = \rho\left(\sum_{i=0}^{l} a_i^\vee \alpha_i^\vee\right) = \sum_{i=0}^{l} a_i^\vee = h^\vee$$

$$\mathfrak{h}^* \longrightarrow \mathring{\mathfrak{h}}^*$$

$$\lambda \longmapsto \bar{\lambda}$$

$$\lambda = \bar{\lambda} + \langle\lambda, K\rangle\Lambda_0 + (\lambda|\Lambda_0)\delta$$

$$\rho = \bar{\rho} + h^\vee\Lambda_0 \qquad |\rho|^2 = |\bar{\rho}|^2$$

- By prop 6.5. $W = \mathring{W}\ltimes\{t_\alpha \mid \alpha\in M\} \simeq \mathring{W}\ltimes M$

$w(\rho)-\rho$? $\qquad ut_\alpha(\rho)-\rho$

$$\lambda\in\mathfrak{h}^* \qquad m := \langle\lambda, K\rangle = \langle\rho, K\rangle = h^\vee$$

$$u\left(t_\alpha(\lambda) = m\Lambda_0 + (\bar{\lambda} + m\alpha) + \frac{1}{2m}(|\lambda|^2 - |\bar{\lambda} + m\alpha|^2)\delta\right)$$

$$t_\alpha(\rho) = h^\vee\Lambda_0 + (\bar{\rho} + h^\vee\alpha) + \frac{1}{2h^\vee}(|\rho|^2 - |\bar{\rho} + m\alpha|^2)\delta$$

$$ut_\alpha(\rho) = \underbrace{u(h^\vee\Lambda_0)}_{0} + u(\bar{\rho}) + u(h^\vee\alpha) + \frac{1}{2h^\vee}(|\rho|^2 - |\bar{\rho} + m\alpha|)u(\delta)$$

$u \in \mathring{W}$ $\quad u\delta = \delta$

$$(12.1.2) \quad ut_\alpha(\rho) - \rho = u(\bar{\rho} + h^\vee\alpha) - \bar{\rho} + \frac{1}{2h^\vee}(|\rho|^2 - |\bar{\rho} + m\alpha|)\delta$$

$$\rho - \bar{\rho} = h^\vee\Lambda_0 \qquad \frac{|\rho|^2}{2h^\vee}\delta$$

$$\overset{ut_\alpha}{\underset{\shortparallel}{w}}(\rho) - \rho$$

$$\sum_{w \in W} \varepsilon(w)\, e(w(\rho) - \rho)$$

$w = u\, t_\alpha$

Hence we obtain (12.13)

$$\text{Right} = e\left(\frac{|\rho|^2}{2h^\vee}\delta\right)\sum_{\alpha \in M}\left(\sum_{w \in \mathring{W}} \varepsilon(w)\, e(w(\bar{\rho} + h^\vee\alpha) - \bar{\rho})\right)$$
$$\times\, e\left(-\frac{1}{2h^\vee}|\bar{\rho} + h^\vee\alpha|^2\delta\right)$$

$$e\left(-\frac{|\rho|^2\delta}{2h^\vee}\right)R^0\prod_{n \geq 1} L(e(-n\delta)) = \sum_{\alpha \in M}\left(\sum_{w \in \mathring{W}} \varepsilon(w)\, e(w(\bar{\rho} + h^\vee\alpha) - \bar{\rho})\right)$$
$$\times\, e\left(-\frac{1}{2h^\vee}|\bar{\rho} + h^\vee\alpha|^2\delta\right)$$

$R^0 = \mathring{\chi}(h^\vee\alpha)$ $\quad 0$

(Macdonald identities) $\quad$ (12.4)

Example :

$A = A_1^{(1)}$

$\alpha_0 \Longleftrightarrow \alpha_1$

$h^\vee = a_0^\vee + a_1^\vee = 2$

$\bar{\rho} = \frac{1}{2}\alpha_1$

$W = \mathring{W} \ltimes \{t_\alpha \mid \alpha \in M\}$ $\quad \mathrm{ran}(A) = l = 1$

$M = \mathring{Q} = \mathbb{Z}\alpha_1$

$\delta = \alpha_0 + \alpha_1 \qquad \forall \alpha \in \Delta \quad \mathrm{mult}(\alpha) = 1$

$\Delta_+ \ni \{ \alpha_1 + m(\underbrace{\alpha_0 + \alpha_1}_{\delta}) \}$

$\Delta_+ \ni \{ -\alpha_1 + m(\delta) \}$

$\Delta_+ \ni \quad n\delta \qquad n \geq 1$

$n\delta - \alpha_0$

$n\alpha_0 + n\alpha_1 - \alpha_0 \qquad \alpha_0 + (n-1)\delta \qquad n\delta$

$\Delta_+ = \{ (n-1)\alpha_0 + n\alpha_1 \; ; \; n\alpha_0 + (n-1)\alpha_1 \; , \; n\alpha_0 + n\alpha_1$

$n = 1, 2 \cdots \}$ $\qquad$ mult $\alpha = 1$ for $\alpha \in \Delta_+$

$\mathring{W} = \{ 1, \; r_{\alpha_1} \}$ $\qquad \mathring{W} = \{\pm 1\}$

$\forall \alpha \in$

the forms of Jacobi triple product identity

$u = e(-\alpha_0) \qquad V = e(-\alpha_1)$

$\Rightarrow$ (12.1.5) $\qquad \prod_{n \geq 1} (1 - u^n v^n)(1 - u^{n-1} v^n)(1 - u^n v^{n-1})$

$$= \sum_{j \in \mathbb{Z}} (-1)^j u^{\frac{1}{2}j(j+1)} v^{\frac{1}{2}j(j-1)}$$

左边 $= \prod_{\alpha \in \Delta_+} (1 - e(-\alpha))^{\mathrm{mult}\,\alpha - 1}$

$$\left(\prod_{n\geq 1}\right)\left(1-e(-n\alpha_0)e(-n\alpha_1)\right)\left(1-e(-(n-1)\alpha_0)e(-n\alpha_1)\right)$$
$$\left(1-e(-n\alpha_0)e(-(n-1)\alpha_1)\right)$$

$$\underline{u=e(-\alpha_0)} \qquad \underline{v=e(-\alpha_1)}$$

$$=\prod_{n\geq 1}(1-u^n v^n)(1-u^{n-1}v^n)(1-u^n v^{n-1})$$

右边: $R_{\text{right}} = e\left(\frac{|\bar{e}|^2}{\underbrace{2h^\vee}_{\frac{1}{8}\delta}}\delta\right)\sum_{j\in\mathbb{Z}}\left(e\left((\bar{e}+h^\vee j\alpha_1)+\varepsilon(\ldots)\right)\right.$

$\boxed{\alpha=j\alpha_1}$

$\cancel{r_{\alpha_1}(\overset{\frac{1}{2}\alpha_1}{\bar{e}}+h^\vee j\alpha_1)-\bar{e}})\Big) \times e\left(-\frac{1}{2h^\vee}|\overset{\frac{1}{2}\alpha_1}{\bar{e}}+h^\vee j\alpha_1|^2\delta\right)$

① $\frac{|\bar{e}|^2}{2h^\vee}=\frac{(\frac{1}{2}\alpha_1|\frac{1}{2}\alpha_1)}{2\cdot 2}=\frac{1}{8}$, $h^\vee=2$ ③ $\frac{1}{8}+j+2j^2$

② $r_{\alpha_1}(\bar{e}+h^\vee j\alpha_1)-\bar{e} = r_{\alpha_1}(\frac{1}{2}\alpha_1+2j\alpha_1)-\frac{1}{2}\alpha_1 = (-\frac{1}{2}-2j)\alpha_1 = (-1-2j)\alpha_1$

$$=e(\cancel{\tfrac{1}{8}})\sum_{j\in\mathbb{Z}}\left(e(\cancel{\bar{e}}+2j\alpha_1-\cancel{\bar{e}})-e((-1-2j)\alpha_1)\right)$$
$$e\left((-\cancel{\tfrac{1}{8}}-j-2j^2)\delta\right)$$

$$=\boxed{\sum_{j\in\mathbb{Z}}\left(e(2j\alpha_1)-e((-1-2j)\alpha_1)\right)e((-j-2j^2)\delta)}$$

$u = e(-\alpha_0) \qquad v = e(-\alpha_1)$

$$= \sum_{j \in \mathbb{Z}} \left(v^{-2j} - v^{2j+1} \right) (uv)^{j+2j^2}$$

$$= \sum_{j \in \mathbb{Z}} u^{j(2j+1)} v^{j(2j-1)} - u^{j(2j+1)} v^{(2j+1)(j+1)}$$

$$= \sum_{k \in 2\mathbb{Z}} u^{\frac{1}{2}k(k+1)} v^{\frac{1}{2}k(k-1)} - \sum_{k \in 2\mathbb{Z}+1} u^{\frac{1}{2}k(k-1)} v^{k\frac{1}{2}(k+1)}$$

$2j = k-1$

$j = \frac{k-1}{2}$

$$= \text{---} - \sum_{-k \in 2\mathbb{Z}+1} u^{\frac{1}{2}(-k)(-k-1)} v^{-k\frac{1}{2}(-k+1)}$$

$$= \text{---} - \sum_{k \in 2\mathbb{Z}+1} u^{\frac{1}{2}k(k+1)} v^{k(\frac{1}{2}(k-1))}$$

$$= \sum_{j \in \mathbb{Z}} (-1)^j u^{\frac{1}{2}k(k+1)} v^{\frac{1}{2}k(k-1)}$$

(12.1.5) #

(12-1-4)

For $\lambda \in \mathfrak{h}^*$ s.t

$$\mathring{\chi}(\lambda) = \frac{\sum_{w \in \mathring{W}} e(\bar{w}(\bar{\lambda}+\bar{\rho}) - \bar{\rho})}{\prod_{\alpha \in \mathring{\Delta}_+} (1 - e(-\alpha))}$$

Note that if $\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_+$ (for $i = 1 \cdots l$)

$\lambda|_{\mathfrak{h}_0^*}$ $\bar{\lambda} \in P_+$

the formal character of the $\mathring{\mathfrak{g}}$ - module $L(\bar{\lambda})$

(12.1.4) $\mathring{R} \longrightarrow$ another form of Macdonald identities:

(12.1.7) $$e\left(-\frac{|\bar{\rho}|^2}{2h^\vee}\delta\right)\prod_{n \geq 1} L(e(-n\delta)) = \sum_{\alpha \in M} \mathring{\chi}(h^\vee \alpha) e\left(-\frac{1}{2h^\vee}|\bar{\rho}+h^\vee\alpha|^2\right)$$

$A = A_l^{(1)}$ $\quad \frac{|\bar{\rho}|}{2h^\vee} = \frac{1}{8}$ $\quad \dim(A_1) = 3$

$$\boxed{\frac{|\bar{\rho}|}{2h^\vee} = \frac{\dim \mathring{\mathfrak{g}}}{24}} \longleftarrow \text{nontwisted affine alg.}$$

"strange" formula

for $X_l^{(1)}$

Setting $q = e(-\delta)$

(12.1.9) $$q^{\frac{\dim \mathring{\mathfrak{g}}}{24}} \prod_{n \geq 1} \left((1-q^n)^l \prod_{\alpha \in \mathring{\Delta}} (1 - q^n e(\alpha))\right) = \sum_{\alpha \in M} \mathring{\chi}(h^\vee \alpha) q^{\frac{|\bar{\rho}+h^\vee\alpha|^2}{2h^\vee}}$$

§12.2.

Let $s = (s_0, s_1, \ldots s_l) = (1, 0 \ldots 0)$ be the basic specialization

we denote by F, $F(e(-\alpha)) = q^{\langle \alpha, d \rangle}$

$\langle \alpha_i, d \rangle = 0$ $\quad (i = 1 \ldots l)$ $\qquad \langle \alpha_0, d \rangle = a_0 = 1$

$F(e(\alpha)) = 1$ if $\alpha \in \mathring{\Delta}$, Lemma 11.11

$F \to p$

We obtain (12.2.2):

$$\left(\mathring{\chi}(\lambda) = \frac{\sum_{w \in \mathring{W}} \varepsilon(w) e(w(\lambda + \bar{\rho}) - \bar{\rho})}{\prod_{\alpha \in \mathring{\Delta}_+} (1 - e(-\alpha))} \right) = \mathring{d}(\lambda) = \prod_{\alpha \in \mathring{\Delta}_+^\vee} \frac{\langle \lambda + \bar{\rho}, \alpha \rangle}{\langle \bar{\rho}, \alpha \rangle}$$

$$p(\mathring{\chi}(\lambda)) = \lim_{q \to 1} \left(\frac{\prod_{\alpha \in \mathring{\Delta}_+} (1 - q^{\langle \lambda + \bar{\rho}, \alpha \rangle})}{\prod_{\alpha \in \mathring{\Delta}_+} 1 - q^{\langle \bar{\rho}, \alpha \rangle}} \right) = \prod_{\alpha \in \mathring{\Delta}_+^\vee} \frac{\langle \lambda + \bar{\rho}, \alpha \rangle}{\langle \bar{\rho}, \alpha \rangle}$$

Eulre product $\varphi(q) = \prod_{n=1}^{\infty} (1 - q^n)$

Dedekind η-function $\eta(q) = q^{1/24} \varphi(q)$ F

F (12.9) $\quad q^{\frac{\dim \mathring{g}}{24}} \prod_{n \geq 1} (1 - q^n)^l \prod_{\alpha \in \mathring{\Delta}} (1 - q^n e(\alpha)) = \sum_{\alpha \in M} \mathring{\chi}(h_\alpha^\vee) q^{\frac{|\bar{\rho} + h_\alpha^\vee|^2}{2h^\vee}}$

$r \searrow$

$$\eta^{\dim \mathring{g}} = \sum_{\alpha \in M} \mathring{d}(h^\vee \alpha)\, q^{\frac{|\bar{e} h^\vee \alpha|}{2h^\vee}}$$

$$\dim \mathring{g} = \underline{l + |\mathring{\Delta}|}$$

$$\| \quad \sum_{\alpha \in \mathring{\Delta}} \dim \mathring{g}_\alpha + \mathfrak{h}$$

$F(e(\alpha)) = 1 \quad \alpha \in \mathring{\Delta}$

$$q^{\frac{\dim \mathring{g}}{24}} \prod_{n \geq 1} (1-q^n)^l \prod (1-q^n)^{|\Delta_0|}$$

$$= q^{\frac{\dim \mathring{g}}{24}} \prod_{n \geq 1} (1-q^n)^{\overbrace{l + |\Delta_0|}^{}}$$

$$= \left(q^{\frac{1}{24}} \prod_{n \geq 1} (1-q^n) \right)^{\dim \mathring{g}}$$

$$\eta(q)$$

$$= \eta(q)^{\dim \mathring{g}}$$

Macdonald η-function identities